D-brane charges and K-homology

Vipul Periwal

Department of Physics, Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

vipul@princeton.edu

Abstract

It is argued that D-brane charge takes values in K-homology. For smooth manifolds with spin structure, this could explain why the phase factor $\Omega(x)$ calculated with a D-brane state x in IIB theory appears in Diaconescu, Moore and Witten's computation of the partition function of IIA string theory.

Diaconescu, Moore and Witten [1] recently compared the partition functions of IIA string theory and M-theory. They found agreement, amusingly interpreted as a derivation of K-theory from M-theory. Many subtle mathematical effects have to conspire for this agreement, so it seems important to explore further the building blocks of their results.

K-theory will enter into the discussion in a variety of ways so here is a summary of K-theory in string theory. Witten [2] argued that D-brane charge should take values in K-theory, following earlier work by Cheung and Yin [5] and Moore and Minasian [4], and providing a mathematical setting for the results of Sen [6]. To be precise, the groups $K^{0(1)}(X)$ are associated with D-branes in IIB(A) string theory on the spacetime X, respectively. In later work, Moore and Witten [3] argued that Ramond-Ramond fields should take values in K-theory as well, with $K^{1(0)}(X)$ classifying these fields in IIB(A) string theory.

An important point in [1] is the computation of a phase factor $\Omega(x)$ associated with an element x which is a Ramond-Ramond field in IIA string theory on a manifold X. Somewhat surprisingly, this phase factor is computed from a D-brane state in IIB string theory, also associated with x by means of the above arguments. The perplexing appearance of IIB string theory in a computation that *a priori* involves only IIA theory is pointed out in footnote 14 in [1].

Regressing a little bit to Polchinski's basic operational definition of D-branes [7], recall that string theory in the presence of a D-brane is naïvely defined by specifying a submanifold M of X and including open strings which have both ends on M. Suppose we act with a diffeomorphism

$$\phi : X \to X', \tag{1}$$

then the submanifold

$$M \mapsto \phi_*(M) \subset X' \tag{2}$$

and this map is covariant, equivalently M is 'pushed-forward' by ϕ_*. On the other hand, if there are forms G defined on X' these fields map contravariantly to define 'pulled-back' forms:

$$\phi^* G(y)(v_1, \ldots, v_n) \equiv \; G(\phi(y))(\phi_* v_1, \ldots, \phi_* v_n) \tag{3}$$

for $v_i(y)$ tangent vectors at y in X, and $\phi_* v_i$ the corresponding pushed-forward vectors.

Now, K-theory is an extraordinary cohomology theory for manifolds, and as such transforms contravariantly, *i.e.* ϕ induces a map

$$\phi^* : K(X') \to K(X). \tag{4}$$

This is perfectly appropriate for Ramond-Ramond fields, but for making contact between the covariant operational definition of D-branes and contravariant K-theory one needs additional structure such as Poincaré duality, as in section 5.1 of [1] for example. Such additional structure is always available when one works on spin$_c$ manifolds for example, so this is not a major assumption in a physical context. However, it is important to keep this assumption in mind since it implies that the correspondence between D-branes, as characterized by Polchinski, and D-brane charges, if associated with elements of $K(X)$, [2] is not canonical.

It is the purpose of this note to explain that Poincaré duality in K-theory makes it more natural for D-brane charges to take values in K-homology groups, where K-homology is the homology theory dual to K-theory. This is to be contrasted to using Poincaré duality in homology-cohomology and then lifting to K-theory. The utility of this point of view is that it will then become apparent why a D-brane *state in IIB string theory* is necessary for defining a phase factor associated with a *Ramond-Ramond field in IIA theory.* Looking a little further, it is to be hoped that computing in an appropriately defined K-homology theory for conformal field theories might lead to some insight into possible D-brane states in an abstract closed string background, perhaps explaining the Cardy conditions [8] as characterizing a cycle in K-homology.

A note on convention: upper indices are contravariant and lower ones covariant. In particular, we have $K_0(C(X)) = K^0(X)$, where $C(X)$ is the algebra of continuous functions on the manifold X.

The basic idea for defining K-homology groups is the following [9–11]: For any elliptic operator D on a manifold X, and a smooth vector bundle E over X, one can define an operator D_E acting on sections of E (by using partitions of unity or a connection on E, for example). The operator D_E depends on the choices but it is a Fredholm operator (*i.e.* has a finite-dimesional kernel and cokernel). Its index does not depend on these choices, so we get a map

$$\mathrm{Index}_D : K^0(X) \to \mathbf{Z} \tag{5}$$

with $\mathrm{Index}_D(E) \equiv \mathrm{Index}(D_E)$. It is crucial in this construction that the operator D is a pseudolocal operator, in other words it commutes with the operation of multiplication by functions on X up to compact operators. Stated algebraically, $K^0(X)$ can be interpreted as equivalence classes of projection operators p on a Hilbert space, and elements of $K_0(X)$ are suitably defined equivalence classes of abstract pseudolocal Fredholm operators F on X.

The natural pairing between these two sets of objects is given by the index map by computing Index(pFp). In topological K-theory expressed in terms of algebras of functions, the definition of K-homology involves classifying extensions of $C(X)$ by the algebra of compact operators up to unitary equivalence [10].

This example serves as the intuition for Kasparov's definition of KK-groups, which seem to be the most natural framework for our purposes. The precise definitions are a little technical [12–15] so all mathematical precision has been eliminated from the following discussion. The tensor products are all $\mathbf{Z}_2$ graded tensor products. $KK^*(A,B)$ is an Abelian group depending contravariantly on the algebra A and covariantly on the algebra B. The elements of $KK^*(A,B)$ are homotopy classes of generalized elliptic operators over A with coefficients in B given by (i) an operator F between two B-modules $\mathcal{E}_{0,1}$ and representations of A as operators on $\mathcal{E}_{0,1}$ such that $[a,F], a(F^*F-1)$ and $a(FF^*-1)$ are all compact operators. The condition $[a,F]$ compact is obviously a generalization of the pseudolocal property in the example above, and the other two conditions require F to be unitary up to compact operators. The facts of interest to us are the following:

$$\text{For } A = \mathbf{C}, \quad KK^*(\mathbf{C}, B) = K_*(B). \tag{6}$$

$$\text{For } B = \mathbf{C}, \quad KK^*(A, \mathbf{C}) = K^*(A). \tag{7}$$

In addition, there is a bilinear associative intersection product

$$KK^n(A_1, B_1 \otimes D) \otimes_D KK^m(D \otimes A_2, B_2) \to KK^{n+m}(A_1 \otimes A_2, B_1 \otimes B_2), \tag{8}$$

with addition mod 2 in the superscript. KK-equivalence of A and B is the statement that the KK groups of $A \otimes E$ and $B \otimes E$ with any other algebras D, E are isomorphic with $A \otimes E$ or $B \otimes E$ as contravariant (resp. covariant) arguments and D as the covariant (resp. contravariant) argument. Finally, $KK(A,A)$ is a ring with unit.

Suppose now that we have two algebras A and B such that there are elements

$$\alpha \in KK(A \otimes B, \mathbf{C}), \quad \beta \in KK(\mathbf{C}, A \otimes B) \tag{9}$$

with the property that

$$\beta \otimes_A \alpha = 1_B \in \; KK(B,B), \beta \otimes_B \alpha = 1_A \in \; KK(A,A). \tag{10}$$

We then get KK-duality isomorphisms

$$K_*(A) \cong K^*(B), \text{ and } K^*(A) \cong K_*(B) \tag{11}$$

between the K-theory(K-homology) of A and the K-homology(K-theory) of B. The algebras A and B are Poincaré dual [14]. In general these algebras are *not* KK-equivalent.

An example of this duality can be constructed as follows [11,14,13]: Suppose that E is the total space of a real vector bundle with a fibre metric over a differentiable manifold X. Consider the algebra of the sections of the complex Clifford bundle $A \equiv \Gamma(\mathrm{Cliff}(E))$ and the algebra $B \equiv C(X)$. Using a partition of unity subordinate to an open cover of X such that E is trivializable over the sets in the open cover, we can define an operator d which restricts to the exterior derivative in the fibre direction locally. d acts on the Hilbert space of square integrable sections of the complexification of $\bigwedge^* E$. Furthermore, $Y \equiv \Gamma(\mathrm{Cliff}(E)) \otimes C(X)$

acts by Clifford multiplication on these sections. We set $F = (d+\delta)/(1+(d+\delta)^2)^{1/2}$, where δ is the adjoint of d. This defines $\alpha \in KK(Y, \mathbf{C})$. For β we note that the algebra of functions on E that vanish at infinity $C_0(E)$ is KK-equivalent to A, so we take the Hilbert module to be Y itself, and construct an element in $KK(\mathbf{C}, C_0(E) \otimes B) \cong KK(\mathbf{C}, Y)$ by taking F to be multiplication by a parametrized version of the Bott element

$$F = v_p(1 + |v_p|_p^2)^{-1/2} \tag{12}$$

for p a point in X, v_p a vector in E_p, with the norm computed with the fibre metric at p.

If E has a spin$_c$ structure, $\Gamma(\mathrm{Cliff}(E))$ is KK-equivalent to $C(X)$ when E has even-dimensional fibres, and KK-equivalent to $C(X)\hat{\otimes}\mathbf{C}_1$ when E has odd-dimensional fibres [13], where $\mathbf{C}_1$ is the Clifford algebra for $\mathbf{C}$, the superalgebra with one generator of odd degree $\sigma : \sigma^2 = 1$. Since KK-groups have the appropriate Clifford periodicity properties, these algebras are KK-equivalent up to the shift for the odd-dimensional case.

For the application to type II string theory, a fairly general context in which we expect IIA and IIB duality is one in which there is a fixed-point free action of $\mathrm{U}(1)^{2n+1}$. This defines a sub-bundle E of TX with fibres of dimension $2n+1$. Then it is natural to conjecture that at the level of KK-theory, the KK-duality with the shift due to the odd dimension of the bundle gives the expected IIA-IIB duality. For strings in ten dimensions, this gives an isomorphism between Ramond-Ramond fields in IIA string theory and D-brane charges in IIB string theory as desired [1]. I want to emphasize that I have *not* shown that this is in fact what we mean by IIA-IIB duality in string theory—I am merely pointing out that this would make sense of the computation in [1], as well as identifying D-brane charge in a way that is consistent with the covariance of the physical description of D-branes [7]. Of course, a U(1) action is not general enough to encompass all manifestations of IIA-IIB duality. The general setting is probably that of Takai duality [13] for $\mathbf{Z}_2$ but with specific constraints so that the Takai dual is also the KK-dual.

To conclude, just considering Ramond-Ramond fields classified by $KK(\mathbf{C}, X)$, or just D-branes classified by $KK(X, \mathbf{C})$, independently may be inadequate for physics. As an example, it seems to me it should be possible to describe the jump in Ramond-Ramond fields across a D-brane either in terms of the local image in cohomology of two different K-theory classes (see [3] for example), or by considering an appropriate element of $KK(X, X)$. This ring has a natural action on both $K^0(X)$ and $K_0(X)$ via the Kasparov product, and it would be interesting to figure out if this action is of physical significance. More concretely, understanding the Cardy conditions defining consistent boundary states as the data defining K-cycles seems worth pursuing.

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REFERENCES

[1] D.-E. Diaconescu, G. Moore and E. Witten, *E_8 gauge theory, and a derivation of K-theory from M-theory*, hep-th/0005090; A derivation of K-theory from M-theory, hep-th/0005091

[2] E. Witten, J. High Energy Phys. 12, 019 (1998)

[3] G. Moore and E. Witten, *Self-duality, Ramond-Ramond fields and K-theory*, hep-th/9912279

[4] R. Minasian and G. Moore, J. High Energy Phys. 11, 002 (1997)

[5] Y.-K. Cheung and Z. Yin, Nucl. Phys. B517, 69 (1998)

[6] A. Sen, J. High Energy Phys. 06, 007 (1998), J. High Energy Phys. 08, 010 (1998), J. High Energy Phys. 08, 012 (1998), J. High Energy Phys. 09, 023 (1998)

[7] J. Polchinski, Phys. Rev. Lett. 75, 4724 (1995)

[8] J. Cardy, Nucl. Phys. B324, 581 (1989)

[9] M.F. Atiyah, Quart. J. Math. Oxford (2) 19, 113 (1968); p. 21 in *Global theory of elliptic operators*, Proc. Int. Symp. Funct. Anal., U. Tokyo Press, Tokyo, 1969

[10] L. Brown, R. Douglas and P. Fillmore, Ann. of Math. 105, 265 (1977)

[11] G. Kasparov, Math USSR Izv. 16, 513 (1981)

[12] N. Higson, *Proc. Symp. Pure Math.* v. 51, Part 1, p. 239, Amer. Math. Soc., 1990

[13] B. Blackader, *K-theory for operator algebras* (second ed.), Cambridge U. Press, 1998

[14] A. Connes, *Noncommutative geometry*, Academic Press, 1994

[15] K. Jensen and K. Thomsen, *Elements of KK-theory*, Birkhäuser, 1991